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Generalized Physical and SRB Measures for Hyperbolic Diffeomorphisms

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In this paper we introduce the notion of generalized physical and SRB measures. These measures naturally generalize classical physical and SRB measures to measures which are supported on invariant sets that are not necessarily attractors. We then perform a detailed case study of these measures for hyperbolic Hénon maps. For this class of systems we are able to develop a complete theory about the existence, uniqueness, finiteness, and properties of these natural measures. Moreover, we derive a classification for the existence of a measure of full dimension. We also consider general hyperbolic surface diffeomorphisms and discuss possible extensions of, as well as the differences to, the results for Hénon maps. Finally, we study the regular dependence of the dimension of the generalized physical/SRB measure on the diffeomorphism. For the proofs we apply various techniques from smooth ergodic theory including the thermodynamic formalism.

KEY WORDS: Natural measures; SRB property; Physical measure; Axiom A; Hénon map; Hausdorff dimension.
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1. INTRODUCTION

1.1. Motivation

It is a long-term goal in dynamical systems to understand the typical dynamics of a given system. Here "typical" usually means for a set of points of full measure with respect to an invariant probability measure. However, for many systems the set of invariant measures is rather large. This raises the question as to which invariant measure is the natural choice to consider. From an applications point of

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view, the only measures which can actually be "observed" are those for which the set of points, whose orbit distribution converges to the measure, has positive volume. These measures are called physical (or sometimes also natural) measures.

To make this mathematically precise, let us consider a $C^{1+\varepsilon}$-diffeomorphism $f : M \to M$ on a smooth d-dimensional Riemannian manifold M, and let Λ be a compact f-invariant set. Denote by $\mathcal{M}$ the space of f-invariant probability measures on Λ endowed with the weak* topology. Moreover, let $\mathcal{M}_E$ denote the subset of ergodic measures. For $\mu \in \mathcal{M}$ we define the basin of μ by

$$\mathcal{B}^+(\mu) = \left\{ x \in M : \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \mu \text{ as } n \to \infty \right\}, \tag{1}$$

where $\delta_{f^i(x)}$ denotes the Dirac measure on $f^i(x)$. Analogously, we denote by $\mathcal{B}^-(\mu)$ the basin of μ with respect to f^{-1}. The basin of μ is sometimes also called the set of future generic points of μ, see [(9)] and [(16)]. A measure $\mu \in \mathcal{M}$ is called a *physical measure* if $\mathcal{B}^+(\mu)$ has positive Lebesgue measure. Moreover, μ is called a *SRB measure* (standing for Sinai, Ruelle, Bowen) if it has at least on positive Lyapunov exponent and its corresponding conditional measures on the unstable manifolds are absolutely continuous with respect to the Lebesgue measure. It turns out that for many systems the notion of physical measures coincides with that of SRB measures. The existence of physical and SRB measures is understood in the case of uniformly hyperbolic systems due to the classical work of Bowen and Ruelle, and also for some non-uniformly hyperbolic systems, in particular for Hénon maps which are small perturbations of one-dimensional maps due to the celebrated work of Benedicks and Carleson [(7)] and Benedicks and Young[(8)]. In all of these cases the set Λ is an attractor. We refer to the expository article[(28)] for more details about physical and SRB measures and further references. Since the basin of any invariant measure must be contained in the stable set $W^s(\Lambda)$ of Λ it is clear that if the Hausdorff dimension $\dim_H W^s(\Lambda)$ of $W^s(\Lambda)$ is strictly smaller than d, then no physical measure exists. This is for example the case when Λ is a uniformly hyperbolic set of a C^2-diffeomorphism which is not an attractor, see[(23)]. However, it is still possible that there exists an invariant measure whose basin is as large as possible. This leads to the following definition. We say that $\mu \in \mathcal{M}_E$ is a *generalized physical measure* if

$$\dim_H \mathcal{B}^+(\mu) = \dim_H W^s(\Lambda). \tag{2}$$

Moreover, we call $\mu \in \mathcal{M}_E$ a *generalized SRB measure* if it has at least one positive Lyapunov exponent and the corresponding conditional measures on the unstable manifolds are absolutely continuous with respect to the t-dimensional Hausdorff measure, where t denotes the Hausdorff dimension of the intersection of the unstable manifolds with Λ. We refer to Section 3 for more details. Obviously, physical and SRB measures are also generalized physical and SRB measures, but the converse is, in general, not true.

Since f is a diffeomorphism, we can also consider the set of points which are generic for μ under forward as well as under backward iteration. We define the two-sided basin of $\mu \in \mathcal{M}$ by $\mathcal{B}(\mu) = \mathcal{B}^+(\mu) \cap \mathcal{B}^-(\mu)$. We say that $\mu \in \mathcal{M}_E$ is a *generalized two-sided physical measure* if $\mathcal{B}(\mu)$ is as large as possible, that is,

$$\dim_H \mathcal{B}(\mu) = \sup_{\nu \in \mathcal{M}_E} \dim_H \mathcal{B}(\nu). \tag{3}$$

We denote by $\dim_H \mu$ the dimension of the measure μ (see Section 2). We note that by Birkhoff's ergodic theorem the basin as well as the two-sided basin of $\mu \in \mathcal{M}_E$ are sets of full measure. Thus, for all $\mu \in \mathcal{M}_E$, we have that

$$\dim_H \mu \leq \dim_H \mathcal{B}(\mu) \leq \dim_H \mathcal{B}^{\pm}(\mu) \cap \Lambda. \tag{4}$$

The main purpose of this paper is to carry out a case study of the existence and uniqueness of the above discussed natural measures in the case of hyperbolic Hénon maps, as well as to discuss possible extensions and differences when considering measures supported on general hyperbolic sets of surface diffeomorphisms. Furthermore, we analyze the regular dependence of the dimension of the natural measures on the diffeomorphism.

We shall now describe our results in the case of hyperbolic Hénon maps. For the corresponding results in the case of general hyperbolic sets on surfaces we refer to Section 6.

1.2. Statement of the Main Results for Hénon Maps

A Hénon map $f = f_{a,b}$ is a diffeomorphism of $\mathbb{R}^2$, which may be written as

$$f(x, y) = (a - x^2 - by, x), \tag{5}$$

where $(a, b) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$. Let $\Lambda \subset \mathbb{R}^2$ denote the set of points with bounded forward and backward orbit. We say that f is hyperbolic if Λ is a hyperbolic set of f. Moreover, we say that f is of maximal entropy if $h_{\text{top}}(f|\Lambda) = \log 2$. We will always assume the f is a hyperbolic Hénon map having maximal entropy. In this case $f|\Lambda$ is topologically conjugate to the full 2-shift, and Λ is a horseshoe of f. In particular, f does not admit a classical physical or SRB measure. Hyperbolic Hénon maps having maximal entropy were discovered by Devaney and Nitecki[10]. We call the corresponding set of parameters the Devaney-Nitecki horseshoe locus. For example, it is known that if $a > 2(1 + |b|)^2$, then $f = f_{a,b}$ belongs to the Devaney–Nitecki horseshoe locus, see[6]. We refer to [6] for more details about the Devaney–Nitecki horseshoe locus. The following result is a consequence of Theorem 8 and Corollary 8 in the text.

Theorem 1. *Let f be a hyperbolic Hénon map having maximal entropy. Then f admits a unique generalized physical measure μ^+, and μ^+ is a Gibbs measure. Moreover, μ^+ is uniquely determined by each of the following properties:*

(i) *μ^+ is the unique generalized SRB measure of f;*
(ii) $\dim_H \mathcal{B}^+(\mu^+) \cap \Lambda = \dim_H \Lambda$;
(iii) $\dim_H \mathcal{B}^+(\mu^+) = \dim_H W^u_\varepsilon(x) \cap \Lambda + 1$.

We note that the analogue of Theorem 1 holds also in the context of general hyperbolic sets on surfaces (see Theorem 15 for the precise statement). Note that f^{-1} is also a hyperbolic Hénon map having maximal entropy. Therefore, Theorem 1 also applies to f^{-1}. We denote the generalized physical measure of f^{-1} by μ^-.

It is well-known that for hyperbolic attractors the equilibrium measure of the potential $-\log|\det Df|E^u|$ is the unique physical respectively SRB measure. Corollary 5 on the other hand shows that for hyperbolic Hénon maps this measure does not coincide with the generalized physical respectively SRB measure. A related result holds in the context of general non-attracting hyperbolic sets of surface diffeomorphisms, namely it is shown in Corollary 10 that the generalized physical respectively SRB measure coincides with the equilibrium measure of the potential $-\log|\det Df|E^u|$ if and only if $\log|\det Df|E^u|$ is cohomologous to a constant. In particular, these measures are distinct for an open and dense set (with respect to the C^1-topology) of surface diffeomorphisms with a non-attracting hyperbolic set (see Corollary 11).

Recall that $\mu \in \mathcal{M}_E$ is an ergodic *measure of maximal dimension* if

$$\dim_H \mu = \sup_{\nu \in \mathcal{M}} \dim_H \nu.$$

These measures have been recently studied by Barreira and Wolf in [(2)]. The next theorem establishes the existence of generalized two-sided physical measures. It compiles results from Corollaries 6 and 8.

Theorem 2. *Let f be a hyperbolic Hénon map having maximal entropy. Then f admits at least one and most finitely many generalized two-sided physical measures. Moreover, μ is a generalized two-sided physical measure if and only if μ is an ergodic measure of maximal dimension.*

We note that a related result holds also in the context of general hyperbolic surface diffeomorphisms (see Theorem 15). However, we are not able anymore to conclude the finiteness of the generalized two-sided physical measures.

So far we have established the existence of different classes of natural measures. The question concerning the relation between these measures is addressed in the next theorem. It combines results from Corollaries 6, 7 as well as Theorems 8, 10, 11 and 12.

Theorem 3. *Let f be a hyperbolic Hénon map having maximal entropy, and let μ^+, μ^- and μ be as in Theorems 1 and 2. Then*

(i) *If f preserves volume, then $\mu = \mu^+ = \mu^-$. Furthermore, μ is the unique ergodic measure of full dimension for f, that is,* $\dim_H \mu = \dim_H \Lambda$.
(ii) *If f is not volume preserving, then μ, μ^+ and μ^- are pairwise disjoint. Moreover, f admits no measure of full dimension.*

To the best of our knowledge Theorem 3 is the first higher-dimensional result which provides a final classification for the existence of a measure of full dimension within a non-trivial family of maps.

Another outcome of this paper is the characterization of the strictness of the inequalities in [4]. Namely, we show that if f is a hyperbolic Hénon map having maximal entropy, then $\dim_H \mu = \dim_H \mathcal{B}(\mu)$ for all $\mu \in \mathcal{M}_E$ (see Theorem 9). On the other hand, $\dim_H \mathcal{B}(\mu) = \dim_H \mathcal{B}^+(\mu) \cap \Lambda$ if and only if μ is the generalized physical measure of f^{-1}. This follows from Theorems 7, 8 and 9.

We note that Theorem 3 (except the simple part (i)) has no immediate extension to the case of general hyperbolic surface diffeomorphisms. The reason for this is that in the case of Hénon maps, part (ii) crucially depends on the a certain cohomology relation which is for general hyperbolic surface diffeomorphisms not satisfied.

The paper is organized as follows. In Section 2 we review some known facts about Hénon maps. Furthermore, we discuss the ergodic and dimension theory of hyperbolic Hénon maps. Section 3 establishes the existence and uniqueness of generalized physical and SRB measures for Hénon maps. In Section 4 we discuss the relation between generalized two-sided physical measures and ergodic measures of maximal dimension in the case of Hénon maps. The existence of measures of maximal and full dimension is analyzed in Section 5. Finally, we study in Section 7 general hyperbolic surface diffeomorphisms and discuss extensions of (and differences to) the results derived for Hénon maps. We also analyze the regular dependence of the dimension of the natural measures on the diffeomorphism.

2. HÉNON MAPS

2.1. Notation and Preliminaries

We start by recalling some standard facts about Hénon maps. Let $f = f_{a,b}$ be a Hénon map (see 5). Let $\Lambda \subset \mathbb{R}^2$ the set of points with bounded forward and backward orbit. It is known that Λ is a compact f-invariant set (see [11]). We define the stable/unstable set of Λ by

$$W^{s/u}(\Lambda) = \{p \in \mathbb{R}^2 : \ \mathrm{dist}(f^{\pm n}(p), \Lambda) \to 0 \text{ for } n \to \infty\}.$$

It follows from the work of [5] that $W^{s/u}(\Lambda)$ is a closed set which coincides with the set of points with bounded forward/backward orbit. Moreover, if $p \in \mathbb{R}^2 \setminus W^{s/u}(\Lambda)$, then the forward/backward orbit of p converges to infinity. The map f^{-1} is given by the formula $f^{-1}(x, y) = (y, b^{-1}(a - y^2 - x))$. Conjugating f^{-1} with $L(x, y) = (by, bx)$, we obtain a map of the form (5). Therefore, f^{-1} is also a Hénon map.

We note that the function $\det Df$ is constant in $\mathbb{R}^2$. Therefore, we can restrict our considerations to the volume decreasing case ($|\det Df| < 1$), and to the volume preserving case ($|\det Df| = 1$), because in the volume increasing case ($|\det Df| > 1$), we can consider f^{-1}.

Let $\mathrm{h_{top}}(f)$ denote the topological entropy of the map $f|\Lambda$. A priori $\mathrm{h_{top}}(f)$ can attain every value in $[0, \log 2]$ (see [11]). Following [5] we say that f is of maximal entropy if $\mathrm{h_{top}}(f) = \log 2$. In this case f has precisely 2^n distinct periodic points with period n all of which are saddle points. Moreover, Λ is a Cantor set which coincides with the closure of the periodic points of f. We say that f is hyperbolic if Λ is a hyperbolic set of f. This means that there exists a continuous Df-invariant splitting $T_\Lambda\mathbb{R}^2 = E^u \oplus E^s$ such that $Df|E^u$ is uniformly expanding and $Df|E^s$ is uniformly contracting. Hyperbolicity implies that we can associate with each point $p \in \Lambda$ its local unstable/stable manifold $W^{u/s}_\varepsilon(p)$. Moreover, we denote by $W^{u/s}(p)$ the global unstable/stable manifolds of p. It follows that f is an Axiom A diffeomorphism and that Λ is the unique basic set of f (see [5] for more details). We would like to point out that there are Axiom A Hénon maps which are not of maximal entropy. For example, in [13] the authors discovered an open set of parameters for which the corresponding Hénon maps are Morse-Smale diffeomorphisms. In particular, these maps are Axiom A, but have zero topological entropy.

Standing Assumptions

We now list several properties which will be assumed in the paper whenever we deal with Hénon maps.

1. $f = f_{a,b}$ is a Hénon diffeomorphism of $\mathbb{R}^2$ with $\mathrm{h_{top}}(f) = \log 2$.
2. Λ is a hyperbolic set of f.
3. f is non-volume increasing.

We recall that assumption 3 is actually not a restriction since we can also consider f^{-1} (see above).

2.2. Ergodic and Dimension Theory for Hénon Maps

We now introduce elements from dimension theory for hyperbolic Hénon maps. We start by introducing Lyapunov exponents. Let $f = f_{a,b}$ be a hyperbolic Hénon map having maximal entropy and let $\mathcal{M}$ be the space of all f-invariant Borel probability measures endowed with weak* topology. Note that $\operatorname{supp} \nu \subset \Lambda$ for every $\nu \in \mathcal{M}$. This makes $\mathcal{M}$ to a compact convex space. Let $\mathcal{M}_E \subset \mathcal{M}$ denote the subspace of ergodic measures. Let $\nu \in \mathcal{M}$. Since Λ is a hyperbolic set of saddle type, there are Lyapunov exponents $\lambda_s(\nu) < 0 < \lambda_u(\nu)$ with respect to ν given by the ν-average of the pointwise Lyapunov exponents. Since, E^u and E^s are one-dimensional, it follows that

$$\lambda_{u/s}(\nu) = \int \log ||Df|E^{u/s}||d\nu. \tag{6}$$

Moreover, since f has constant Jacobian determinant, we have that

$$\lambda_s(\nu) = -\lambda_u(\nu) + \log|\det Df|. \tag{7}$$

We now discuss the Hausdorff dimension of a measure ν. Recall that the Hausdorff dimension (or simply the dimension) of a measure $\nu \in \mathcal{M}$ is defined by

$$\dim_H \nu = \inf\{\dim_H A : \nu(A) = 1\}, \tag{8}$$

where $\dim_H A$ denotes the Hausdorff dimension of the set. In general the dimension of ν is strictly smaller than the dimension of its support. We say that ν is a *measure of full dimension* if $\dim_H \nu = \dim_H \Lambda$.

If ν is ergodic, then by Young's formula [26], we have,

$$\dim_H \nu = \frac{h_\nu(f)}{\lambda_u(\nu)} + \frac{h_\nu(f)}{\lambda_u(\nu) - \log|\det Df|}, \tag{9}$$

where $h_\nu(f)$ denotes the measure theoretic entropy of f with respect to ν.

In general, we will deal with measures which are not necessarily ergodic. However, the following result due to Barreira and Wolf (see [3]) often allows us to restrict our considerations to the case of ergodic measures.

Theorem 4. *Let $\nu \in \mathcal{M}$. Then*

$$\dim_H \nu = \operatorname{esssup}\{\dim_H \mathrm{m} : \mathrm{m} \in \mathcal{M}\},$$

where the essential supremum can be taken with respect to any ergodic decomposition τ of ν.

It is a consequence of Theorem 4 (see [3] for more details) that

$$\delta(f) \stackrel{\text{def}}{=} \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} = \sup\{\dim_H \nu : \nu \in \mathcal{M}\}. \tag{10}$$

If a measure μ attains the supremum on the right-hand side of (10), that is $\dim_H \mu = \delta(f)$, we say that μ is a *measure of maximal dimension*. We refer to the article [2] for a detailed discussion of the existence and ergodicity of measures of maximal dimension.

Next we introduce topological pressure. Let $C(\Lambda)$ denote the Banach space of all continuous real valued functions on Λ. In analogy to physics we call $C(\Lambda)$ the space of potentials. The topological pressure of $f|\Lambda$, denoted by $P = P(f|\Lambda, .)$, is a mapping from $C(\Lambda)$ to $\mathbb{R}$ (see [18] for the definition). The variational principle gives the formula

$$P(\varphi) = \sup_{\nu \in \mathcal{M}} \left(h_\nu(f) + \int \varphi d\nu \right). \tag{11}$$

If a measure $\nu_\varphi \in \mathcal{M}$ achieves the supremum in Eq. (11), that is,

$$P(\varphi) = h_{\nu_\varphi}(f) + \int \varphi d\nu_\varphi, \tag{12}$$

we call it an equilibrium measure of the potential φ.

We recall that two functions $\varphi, \psi \colon \Lambda \to \mathbb{R}$ are said to be cohomologous if $\varphi - \psi = \eta - \eta \circ f$ for some continuous function $\eta \colon \Lambda \to \mathbb{R}$. In this case we have $P(\psi) = P(\varphi)$. Given $\alpha \in (0, 1]$, let $C^\alpha(\Lambda)$ denote the space of Hölder continuous functions $\varphi \colon \Lambda \to \mathbb{R}$ with Hölder exponent α. We now list several properties of the topological pressure which are needed later on (see [18] for details). Let $\alpha \in (0, 1]$ be fixed. Then:

1. The map $\varphi \mapsto P(\varphi)$ is convex, and when restricted to $C^\alpha(\Lambda)$, it is real-analytic;
2. Each function $\varphi \in C^\alpha(\Lambda)$ has a unique equilibrium measure $\nu_\varphi \in \mathcal{M}$. Furthermore ν_φ is ergodic and given $\psi \in C^\alpha(\Lambda)$ we have,
$$\frac{d}{dt} P(\varphi + t\psi)\Big|_{t=0} = \int_\Lambda \psi \, d\nu_\varphi; \tag{13}$$
3. For each $\varphi, \psi \in C^\alpha(\Lambda)$ we have that $\nu_\varphi = \nu_\psi$ if and only if $\varphi - \psi$ is cohomologous to a constant;
4. For each $\varphi, \psi \in C^\alpha(\Lambda)$ and $t \in \mathbb{R}$ we have,
$$\frac{d^2}{dt^2} P(\varphi + t\psi) \geq 0, \tag{14}$$
with equality if and only if ψ is cohomologous to a constant.

We have $P(0) = h_{\text{top}}(f) = \log 2$. It follows from the properties above that ν_0 is the unique measure of maximal entropy of f. We now introduce potentials which are related to the Lyapunov exponents. We define

$$\phi_{u/s} : \Lambda \to \mathbb{R}, \qquad p \mapsto \log ||Df(p)|E_p^{u/s}|| \tag{15}$$

and the unstable/stable pressure function $P^{u/s} : \mathbb{R} \to \mathbb{R}$ by

$$P^u(t) = P(-t\phi_u) \quad \text{and} \quad P^s(t) = P(t\phi_s). \tag{16}$$

Obviously, the functions $-\phi^u$ and ϕ_s are strictly negative. This implies that $P^{u/s}$ is strictly decreasing.

Since $\phi_{u/s}$ is Hölder continuous (see [(1)]), property 1 of the topological pressure implies that $P^{u/s}$ is real analytic. Property 2 of the topological pressure implies that there exist uniquely defined equilibrium measures $\nu_{\mp t\phi_{u/s}} \in \mathcal{M}$ of the potentials $\mp t\phi_{u/s}$.

Let $\mathcal{S}$ denote the set of all saddle points of f. Note that $\Lambda = \overline{\mathcal{S}}$ (see [(5)]). For $p \in \mathcal{S}$ with period n we denote by $\lambda_{u/s}(p)$ the eigenvalues of $Df^n(p)$, where $|\lambda_s(p)| < 1 < |\lambda_u(p)|$. Hence,

$$\lambda_u(p)\lambda_s(p) = \det Df^n. \tag{17}$$

Using (17) and applying Proposition 4.5 of [(1)], we obtain the following.

Proposition 1. *Let $t \in \mathbb{R}$. Then*

(i) $P^u(t) = P^s(t) - t \log|\det Df|$;
(ii) $\nu_{-t\phi^u} = \nu_{t\phi^s}$.

We will use in the remainder of this paper the notation $\nu_t = \nu_{\mp t\phi^{u/s}}$. This notation is justified by Proposition 1. We also write $\lambda_{u/s}(t) = \lambda_{u/s}(\nu_t)$ and $h(t) = h_{\nu_t}(f)$, and consider $\lambda_{u/s}$ and h as real-valued functions of t. Equations (6), (12) imply

$$P^u(t) = h(t) - t\lambda_u(t). \tag{18}$$

Therefore, by Proposition 4,

$$P^s(t) = h(t) - t(\lambda_u(t) - \log|\det Df|). \tag{19}$$

We will now prove a cohomology property which will be crucial for our results about Hénon maps.

Proposition 2. *Let $f = f_{a,b}$ be a hyperbolic Hénon map having maximal entropy. Then the potential ϕ_u is not cohomologous to a constant.*

Proof: Assume on the contrary that ϕ_u is cohomologous to a constant. Then, by Proposition 4.5 of [(1)], there is $K > 0$ such that

$$|\lambda_u(p)|^{\frac{1}{n(p)}} = K \tag{20}$$

for every periodic point $p \in \mathcal{S}$ with prime period $n(p)$. We will show that (20) does not hold for all possible parameters (a, b). First we consider the two fixed points of f. Set

$$\alpha_{1/2} = \frac{-(b+1) \pm \sqrt{(b+1)^2 + 4a}}{2}. \tag{21}$$

Evidently, the points $p = (\alpha_1, \alpha_1)$ and $q = (\alpha_2, \alpha_2)$ are the fixed points of f. Since $h_{top}(f) = \log 2$ (f is of maximal entropy), it follows from [5] that that $p, q \in \mathbb{R}^2$ and $p \neq q$. Hence,

$$a > -(b+1)^2/4. \tag{22}$$

It is easy to see that $\lambda_{1/2}(p) = -\alpha_1 \pm \sqrt{\alpha_1 - b}$ and $\lambda_{1/2}(q) = -\alpha_2 \pm \sqrt{\alpha_2 - b}$ are the eigenvalues of $Df(p)$ and $Df(q)$ respectively. We claim that

$$|\lambda_u(p)| = |\alpha_1| + \sqrt{\alpha_1^2 - b}. \tag{23}$$

It suffices to show that $\alpha_1^2 > b$, because in this case $\lambda_u(p)$ is real. Assume that $\alpha_1^2 \leq b$. In this case it follows that $|\lambda_1(p)| = |\lambda_2(p)|$. Since p is a saddle point, this is impossible, which proves the claim. Analogously, we can show that

$$|\lambda_u(q)| = |\alpha_2| + \sqrt{\alpha_2^2 - b}. \tag{24}$$

First, we consider the case $|\alpha_1| \neq |\alpha_2|$. In this case it follows from (23) and (24) that $|\lambda_u(p)| \neq |\lambda_u(q)|$. Therefore, (20) does not hold for p and q. Next, we consider the case $|\alpha_1| = |\alpha_2|$. Since $p \neq q$, we must have $\alpha_1 = -\alpha_2$. Therefore, (21) implies $b = -1$. Assume from now on that $b = -1$. Hence, $p = (\sqrt{a}, \sqrt{a})$ and $|\lambda_u(p)| = \sqrt{a} + \sqrt{a+1}$. Obviously, $r = (\sqrt{a}, -\sqrt{a})$ is a periodic point of f of period 2. Note that

$$Df^2(r) = \begin{pmatrix} 1-4a & 2\sqrt{a} \\ -2\sqrt{a} & 1 \end{pmatrix}. \tag{25}$$

It follows that $\lambda_{1/2}(r) = 1 - 2a \pm 2\sqrt{a(a-1)}$ are the eigenvalues of $Df^2(r)$. The fact that r is a saddle point (also using 22) implies that $a > 1$. Hence,

$$|\lambda_u(r)| = 2a - 1 + 2\sqrt{a(a-1)}. \tag{26}$$

Assume that (20) holds for p and r. Then

$$\lambda_u(p)^2 = |\lambda_u(r)|. \tag{27}$$

But this would imply that

$$\sqrt{a(a-1)} - \sqrt{a(a+1)} = 1, \tag{28}$$

which is impossible. We conclude that (20) is not true in the case $b = -1$, which completes the proof. □

Next, we consider the functions λ_u and h.

Lemma 1. *The functions λ_u and h are real analytic. Furthermore, if $t_0 \geq 0$, then*

$$\frac{d\lambda_u}{dt}(t_0) < 0. \tag{29}$$

In particular, λ_u is strictly decreasing.

Proof: Let $t_0 \geq 0$ and ϕ_u as in (15). We define potentials $\varphi = -t_0\phi_u$, $\psi = -\phi_u$ (here we use the notation of (13)). Therefore, application of Equations (6) and (13) implies

$$\frac{dP^u}{dt}(t_0) = -\lambda_u(\nu_{t_0}) = -\lambda_u(t_0). \tag{30}$$

Since P^u is real analytic, we obtain that λ_u is also real analytic. Thus, by (18), h is also real-analytic. Finally, using (14) and Proposition 2, we conclude that

$$\frac{d^2P^u}{dt^2}(t_0) > 0. \tag{31}$$

Hence

$$\frac{d\lambda_u}{dt}(t_0) < 0. \tag{32}$$

□

Hausdorff dimensions of the measures ν_t. We use the notation $\Delta(t) = \dim_H \nu_t$. Equation (9) yields

$$\Delta(t) = \frac{h(t)}{\lambda_u(t)} + \frac{h(t)}{\lambda_u(t) - \log|\det Df|}. \tag{33}$$

We conclude that Δ is also a real analytic function. Equations (18), (19) and Proposition 1 imply

$$\Delta(t) = 2t + \frac{P^u(t)}{\lambda_u(t)} + \frac{P^u(t) + t\log|\det Df|}{\lambda_u(t) - \log|\det Df|}. \tag{34}$$

An elementary calculation gives the following formula for the derivative of Δ:

$$\frac{d\Delta}{dt}(t_0) = -\frac{\frac{d\lambda_u}{dt}(t_0)\,[A(t_0) + B(t_0)]}{\lambda_u(t_0)^2(\lambda_u(t_0) - \log|\det Df|)^2}, \tag{35}$$

where

$$A(t_0) = P^u(t_0)(\lambda_u(t_0) - \log|\det Df|)^2 \tag{36}$$

and

$$B(t_0) = (P^u(t_0) + t_0 \log|\det Df|)\lambda_u(t_0)^2. \tag{37}$$

Hausdorff dimension of the invariant sets. First, we consider the Hausdorff dimension of Λ. The following result is a consequence of work of McCluskey and Manning [17]. It provides a formula for the Hausdorff dimension of the unstable/stable slice in terms of the zeros of the pressure functions.

Theorem 5. *Let f be a hyperbolic Hénon map having maximal entropy. Then* $t^{u/s} = \dim_H W_\varepsilon^{u/s}(x) \cap \Lambda$ *does not depend on* $x \in \Lambda$. *Furthermore,* $t^{u/s}$ *is given by the unique solution of*

$$P^{u/s}(t) = 0, \tag{38}$$

and

$$\dim_H \Lambda = t^u + t^s. \tag{39}$$

Equation (38) is called the Bowen-Ruelle formula. We refer to $t^{u/s}$ as the Hausdorff dimension of the unstable/stable slice. As a consequence of Theorem 5 we obtain an inequality between t^u and t^s.

Corollary 1. *We have* $t^s, t^u < 1$. *Moreover,*

$$t^s \leq \frac{t^u \log 2}{\log 2 - t^u \log|\det Df|}. \tag{40}$$

In particular, if f is volume-decreasing, then $t^s < t^u$.

Proof: That $t^{u/s} < 1$ follows from $P(\mp\phi^{u/s}) < 0$ (see [1]) and Theorem 5. Recall that f is non-volume increasing. We have $P^s(0) = \log 2$. Moreover, by Proposition 4 and Theorem 5 we have, $P^s(t^u) = t^u \log|\det Df|$. Therefore, since P^s is a convex function, the graph of P^s lies below the line segment joining $(0, \log 2)$ and $(t^u, t^u \log|\det Df|)$. The result follows from Theorem 5. □

It is a consequence of Theorem 5, the variational principle and the uniqueness of the equilibrium measure of a Hölder continuous potential that

$$\dim_H \Lambda = t^u + t^s = \sup_{\nu\in\mathcal{M}} \left(\frac{h_\nu(f)}{\lambda_u(\nu)}\right) + \sup_{\nu\in\mathcal{M}} \left(-\frac{h_\nu(f)}{\lambda_s(\nu)}\right), \tag{41}$$

where each of the suprema on the right-hand side of the equation is uniquely attained by the measures ν_{t^u} and ν_{t^s} respectively. Hence

$$\dim_H \Lambda = \frac{h(t^u)}{\lambda_u(t^u)} - \frac{h(t^s)}{\lambda_s(t^s)}. \tag{42}$$

Equation (42) and the uniqueness of the measures ν_{t^u}, ν_{t^s} in Equation (41) imply that, if there exists an ergodic measure μ of full dimension, then

$$\mu = \nu_{t^u} = \nu_{t^s}. \tag{43}$$

Thus, we have the following result.

Corollary 2. *Assume μ is an ergodic measure of full dimension for f, then $\mu = \nu_{t^u} = \nu_{t^s}$. In particular, there exists at most one ergodic measure of full dimension.*

Remark. It should be noted that without the assumption of ergodicity, if a measure of full dimension exists, then it is never unique. For example, if μ is an ergodic measure of full dimension for *f*, then every measure $\nu \in \mathcal{M}$, which has an ergodic decomposition that puts positive measure on μ, is a measure of full dimension. This follows from Theorem 4.

Next, we consider the Hausdorff dimension of the stable and unstable set of Λ.

Theorem 6. $\dim_H W^{s/u}(\Lambda) = t^{u/s} + 1$.

Proof: The proof is similar to the corresponding proof for polynomial automorphisms of $\mathbb{C}^2$ (see Theorem 4.1 in [(29)]). Therefore, we provide only a sketch. Without loss of generality we only consider $W^s(\Lambda)$. It follows from the Shadowing Lemma that

$$W^s(\Lambda) = \bigcup_{p \in \Lambda} W^s(p). \tag{44}$$

Since Λ has a local product structure, it suffices to prove that there is $\varepsilon > 0$ such that

$$\dim_H \Big(\bigcup_{q \in W^u_\varepsilon(p) \cap \Lambda} W^s_\varepsilon(q) \Big) = t^u + 1 \tag{45}$$

for all $p \in \Lambda$. We refer to [(29)] for more details. Let $p \in \Lambda$. Set $A_p = W^u_\varepsilon(p) \cap \Lambda$. Following [(29)], for $\varepsilon > 0$ small enough, there exists a homeomorphism

$$h : A_p \times (-\varepsilon, +\varepsilon) \to \bigcup_{q \in A_p} W^s_\varepsilon(q), \tag{46}$$

with the property that $h(q \times (-\varepsilon, +\varepsilon)) = W^s_\varepsilon(q)$ for all $q \in A_p$. Moreover, *h* and h^{-1} are α-Hölder continuous for every $\alpha \in (0, 1)$. This implies (45). □

3. GENERALIZED PHYSICAL AND SRB MEASURES

In this Section we develop the theory of generalized physical and SRB measures for hyperbolic Hénon maps. We start with some definitions.

Given $p \in \mathbb{R}^2$, we consider a sequence of probability measures $(\nu_n(p))_{n\in\mathbb{N}}$ defined by

$$\nu_n(p) = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(p)}.$$

We would like to define what it means that $\nu_n(p) \to \nu \in \mathcal{M}$ for $n \to \infty$ also for points $p \in \mathbb{R}^2 \setminus \Lambda$. This can be done in the following way: Let U be a neighborhood of Λ. Fix for each $\phi \in C(\Lambda)$ a continuous extension $\widehat{\phi} : U \to \mathbb{R}$ of ϕ. We say that $(\nu_n(p))_{n\in\mathbb{N}}$ converges to $\nu \in \mathcal{M}$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \widehat{\phi}(f^i(p)) = \int \phi d\nu \tag{47}$$

for all $\phi \in C(\Lambda)$. Here we use the convention $\widehat{\phi}(f^i(p)) = 0$ if $f^i(p) \notin U$. It can be shown that this convergence does not depend on the choice of the extensions $\widehat{\phi}$. Evidently, it is necessary for $\lim_{n\to\infty} \nu_n = \nu$ that $p \in W^s(\Lambda)$, because otherwise $\lim_{n\to\infty} f^n(p) = \infty$.

It follows from Birkhoff's ergodic theorem that if $\nu \in \mathcal{M}_E$ then $\nu(\mathcal{B}^+(\nu)) = 1$ and if $\nu \in \mathcal{M} \setminus \mathcal{M}_E$ then $\nu(\mathcal{B}^+(\nu)) = 0$. This is the reason why we will always require that our measures are ergodic.

Let $\nu \in \mathcal{M}$. Then, for ν-almost every $p \in \Lambda$, there exists a conditional measure ν_p supported on $W^u_\varepsilon(p)$ such that the system $\{\nu_p : p \in \Lambda\}$ de-integrates ν. More precisely,

$$\int \phi d\nu = \int_\Lambda \left(\int \phi d\nu_p \right) d\nu \tag{48}$$

for all $\phi \in C(\Lambda)$. We refer to [27] for more details.

Definition 1. Let f be a hyperbolic Hénon map having maximal entropy. We say that $\mu \in \mathcal{M}_E$ is a generalized physical measure for f if $\dim_H \mathcal{B}^+(\mu) = \dim_H W^s(\Lambda)$. Moreover, we call $\mu \in \mathcal{M}_E$ a generalized SRB-measure for f if for μ-almost every $p \in \Lambda$ the corresponding conditional measure μ_p is absolute continuous with respect to the t-dimensional Hausdorff measure, where $t = \dim_H W^u_\varepsilon(p) \cap \Lambda$.

Remarks.

(i) We note that in the definition of the basin $\mathcal{B}^+(\mu)$ we include points $p \in \mathbb{R}^2$ being future generic for μ in the sense of (47).

(ii) The definitions above include the case of classical physical and SRB measures. In this case the t-dimensional Hausdorff measure is equivalent to the Lebesgue measure.

In order to analyze generalized physical and SRB measures we now develop formulas for the dimension of the basins. We start with an elementary Lemma.

Lemma 2. *Let f be a hyperbolic Hénon map having maximal entropy. Let $\mu \in \mathcal{M}_E$ and let $p \in \mathbb{R}^2$. Then $p \in \mathcal{B}^+(\mu)$ if and only if $p \in W^s(q)$ for some $q \in \Lambda \cap \mathcal{B}^+(\mu)$.*

Proof: Let $p \in \mathcal{B}^+(\mu)$, in particular $p \in W^s(\Lambda)$. Thus, (44) implies that $p \in W^s(q)$ for some $q \in \Lambda$. We conclude that there exist constants $c > 0$ and $0 < \gamma < 1$ such that

$$|f^n(p) - f^n(q)| < c\gamma^n \tag{49}$$

for all $n \in \mathbb{N}$. Let $\psi \in C(\Lambda, \mathbb{R})$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} &\left|\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k(q)) - \int \psi d\mu\right| \\ &\leq \left|\frac{1}{n}\sum_{k=0}^{n-1}\psi(f^k(q)) - \frac{1}{n}\sum_{k=0}^{n-1}\widehat{\psi}(f^k(p))\right| + \left|\frac{1}{n}\sum_{k=0}^{n-1}\widehat{\psi}(f^k(p)) - \int \psi d\mu\right| \end{aligned} \tag{50}$$

Using that $p \in \mathcal{B}^+(\mu)$ and (49), we may conclude that both of the terms in (50) converge to 0 if $n \to \infty$. Hence $q \in \mathcal{B}^+(\mu)$. Conversely, let $p \in W^s(q)$ for some $q \in \Lambda \cap \mathcal{B}^+(\mu)$. Analogously to (50), we can show that $q \in \mathcal{B}^+(\mu)$. □

Recall that $t^{u/s}$ denotes the Hausdorff dimension of the unstable/stable slice. We will now prove two dimension formulas for the basin of an ergodic measure.

Theorem 7. *Let f be a hyperbolic Hénon map having maximal entropy, and let $\nu \in \mathcal{M}_E$. Then*

$$\dim_H \mathcal{B}^+(\nu) = \frac{h_\nu(f)}{\lambda_u(\nu)} + 1 \tag{51}$$

and

$$\dim_H \mathcal{B}^+(\nu) \cap \Lambda = \frac{h_\nu(f)}{\lambda_u(\nu)} + t^s. \tag{52}$$

Proof: First, we prove (51). It follows from Lemma (2) and (44) that

$$\mathcal{B}^+(\nu) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(\bigcup_{p\in\mathcal{B}^+(\nu)\cap\Lambda} W^s_\varepsilon(p)\right). \tag{53}$$

Since Λ is a compact set with a local product structure, it is sufficient to show that if $\varepsilon > 0$ is small enough, then for each $p \in \Lambda$,

$$\dim_H\left(\bigcup_{q\in A_p} W^s_\varepsilon(q)\right) = \frac{h_\nu(f)}{\lambda_u(\nu)} + 1, \tag{54}$$

where $A_p = W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu)$. Let $p \in \Lambda$ and let $\varepsilon > 0$ be small. It is a result of Manning[15] that

$$\dim_H A_p = \frac{h_\nu(f)}{\lambda_u(\nu)}. \tag{55}$$

Therefore, (54) can be shown analogously as Theorem 6 by using the α-Hölder homeomorphism h. Next we prove (52). Let $p \in \Lambda$ and $\varepsilon > 0$ small. Since Λ has a local product structure, the map

$$P : W^u_\varepsilon(p) \cap \Lambda \times W^s_\varepsilon(p) \cap \Lambda \to \Lambda, \quad (q, r) \mapsto W^s_\varepsilon(q) \cap W^u_\varepsilon(r) \tag{56}$$

is a well-defined homeomorphism onto a neighborhood U_p of p in Λ. Since the holonomy maps are Lipschitz continuous, (see [14]), it follows that P is a bi-Lipschitz map. In particular, P preserves Hausdorff dimension. We claim that

$$P^{-1}(U_p \cap \mathcal{B}^+(\nu)) = W^u_\varepsilon(p) \cap \mathcal{B}(\nu) \times W^s_\varepsilon(p) \cap \Lambda. \tag{57}$$

Let $(q, r) \in P^{-1}(U_p \cap \mathcal{B}^+(\nu))$. Thus, $q \in W^u_\varepsilon(p) \cap \Lambda, r \in W^s_\varepsilon(p) \cap \Lambda$ and $P(q, r) = W^s_\varepsilon(q) \cap W^u_\varepsilon(r) \in \Lambda \cap \mathcal{B}^+(\nu)$. Applying Lemma 2, we obtain that $q \in \mathcal{B}^+(\nu)$. This implies that $(q, r) \in W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu) \times W^s_\varepsilon(p) \cap \Lambda$. Conversely, let $(q, r) \in W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu) \times W^s_\varepsilon(p) \cap \Lambda$. The fact that $\mathcal{B}^+(\nu) \subset W^s(\Lambda)$ implies $q \in \Lambda$. By (56), $P(q, r) \in W^s_\varepsilon(q)$, and since $q \in \mathcal{B}^+(\nu)$, we deduce from Lemma 2 that $P(q, r) \in \mathcal{B}^+(\nu)$, which proves the claim.
Since P preserves Hausdorff dimension, we may conclude that

$$\dim_H U_p \cap \mathcal{B}^+(\nu) = \dim_H \left(W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu) \times W^s_\varepsilon(p) \cap \Lambda\right). \tag{58}$$

On the other hand, it is well-known that $\dim_H W^s_\varepsilon(p) \cap \Lambda = \overline{\dim}_B W^s_\varepsilon(p) \cap \Lambda$ (see for instance [19]), where $\overline{\dim}_B$ denotes the upper box dimension of the set. Therefore,

$$\begin{aligned} &\dim_H \left(W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu) \times W^s_\varepsilon(p) \cap \Lambda\right) \\ &\quad = \dim_H W^u_\varepsilon(p) \cap \mathcal{B}^+(\nu) + \dim_H W^s_\varepsilon(p) \cap \Lambda. \end{aligned} \tag{59}$$

Combining (55), (58), (59) with a compactness argument completes the proof of the theorem. □

Applying Theorem 7 to f^{-1} gives the following.

Corollary 3. *Let f be a hyperbolic Hénon map having maximal entropy, and let $\nu \in \mathcal{M}_E$. Then*

$$\dim_H \mathcal{B}^-(\nu) = -\frac{h_\nu(f)}{\lambda_s(\nu)} + 1 \tag{60}$$

and

$$\dim_H \mathcal{B}^-(\nu) \cap \Lambda = -\frac{h_\nu(f)}{\lambda_s(\nu)} + t^u. \tag{61}$$

The following result establishes the existence and uniqueness of the generalized physical measure.

Theorem 8. *Let f be a hyperbolic Hénon map having maximal entropy, and let $\mu \in \mathcal{M}_E$. Then the following are equivalent:*

(i) $\mu = \nu_{t^u}$;
(ii) *μ is a generalized physical measure for f;*
(iii) *μ is a generalized SRB measure for f;*
(iv) $\dim_H \mathcal{B}^+(\mu) \cap \Lambda = \dim_H \Lambda$.

Proof: (i) ⇒ (ii) Assume $\mu = \nu_{t^u}$. Thus, by (41), $t^u = h_\mu(f)/\lambda_u(\mu)$. Therefore, Theorems 6 and 7 imply that $\dim_H \mathcal{B}^+(\mu) = t^u + 1 = \dim_H W^s(\Lambda)$, which is (ii). (ii) ⇒ (i) Assume μ is a generalized physical measure. Then, Theorem 6 and (51) imply $t^u = h_\mu(f)/\lambda_u(f)$, and thus (41) implies $\mu = \nu_{t^u}$. (i)⇒ (iii) is well-known, see for instance Theorem 22.1 of [19]. (iii)⇒ (iv) Assume μ is a generalized SRB measure for f. Then, the corresponding conditional measures μ_p supported on the unstable manifolds have Hausdorff dimension t^u for μ-almost every $p \in \Lambda$. Therefore, by (39) and (52), $\dim_H \mathcal{B}^+(\mu) \cap \Lambda = \dim_H \Lambda$. (iv)⇒ (i) If (iv) holds, then by (39) and (52), $t^u = h_\mu(f)/\lambda_u(f)$. Therefore, (41) implies $\mu = \nu_{t^u}$. □

We now list two immediate consequences of Theorem 8.

Corollary 4. *Let f be as in Theorem 8 and let μ^+ be the generalized physical resp. SRB measure for f. Then μ^+ is a Gibbs measure.*

Proof: Since μ^+ is the equilibrium measure of the Hölder continuous potential, $-t^u\phi_u$ it is a Gibbs measure (see [1]). □

Corollary 5. *Let f be as in Theorem 8 and let μ^+ be the generalized physical resp. SRB measure for f. Then μ^+ is distinct from the equilibrium measure of the potential $-\phi_u$, that is $\mu^+ \neq \nu_1$.*

Proof: Since $t^u < 1$ (see Corollary 5), Lemma 1 implies that $\nu_{t^u} \neq \nu_1$. □

Remark. We note that in the case of hyperbolic attractors the equilibrium measure of the potential $-\phi_u$ is the unique SRB resp. physical measure. Therefore, Corollary 5 indicates a difference between the theory of hyperbolic attractors and hyperbolic non-attracting sets. It also follows that our concept of generalized SRB measures differs from that considered by Ruelle in [22].

4. GENERALIZED TWO-SIDED PHYSICAL MEASURES AND MEASURES OF MAXIMAL DIMENSION

In this section we show that for hyperbolic Hénon maps generalized two-sided physical measures are measures of maximal dimension and vice versa. We start by giving the definition of generalized two-sided physical measures. Recall that the two-sided basin of a measure $\mu \in \mathcal{M}$ is defined by $\mathcal{B}(\mu) = \mathcal{B}^+(\mu) \cap \mathcal{B}^-(\mu)$. It follows that $\mathcal{B}(\mu) \subset W^s(\Lambda) \cap W^u(\Lambda)$. Hence $\mathcal{B}(\mu) \subset \Lambda$.

Definition 2. Let f be a hyperbolic Hénon map having maximal entropy. We say that $\mu \in \mathcal{M}_E$ is a generalized two-sided physical measure if

$$\dim_H \mathcal{B}(\mu) = \sup_{\nu \in \mathcal{M}_E} \dim_H \mathcal{B}(\nu). \tag{62}$$

We now prove the main result of this section.

Theorem 9. *Let f be a hyperbolic Hénon map having maximal entropy, and let $\nu \in \mathcal{M}_E$. Then* $\dim_H \mathcal{B}(\nu) = \dim_H \nu$.

Proof: Let $\nu \in \mathcal{M}_E$. It follows from Birkhoff's ergodic theorem (applied to f and f^{-1}) that $\nu(\mathcal{B}(\nu)) = 1$. Therefore, $\dim_H \nu \leq \dim_H \mathcal{B}(\nu)$ is a consequence of the definition of the dimension of the measure ν. To show the reverse inequality we define for $p \in \Lambda$ the pointwise dimension of ν at x by

$$d_\nu(p) = \lim_{r \to 0} \frac{\log \nu(B(p, r))}{\log r} \tag{63}$$

whenever the limit exists. It follows from work of Young [26] (see [19] for a detailed discussion) that if $p \in \mathcal{B}(\nu)$ then $d_\nu(p)$ exists and $d_\nu(p) = \dim_H \nu$. Therefore, $\dim_H \mathcal{B}(\nu) \leq \dim_H \nu$ is a consequence of Theorem 7.2 in [19]. □

Remark. We note that there is an alternative way to prove Theorem 9. Namely, one can combine Manning's formulas for $W_\varepsilon^{u/s}(p) \cap \mathcal{B}^{\pm}(\nu)$ (see [16]) with the fact that Λ has a bi-Lipschitz continuous product structure.

The following is an immediate consequence of Theorem 9 and (10).

Corollary 6. *Let f be a hyperbolic Hénon map having maximal entropy, and let $\mu \in \mathcal{M}_E$. Then μ is a generalized two-sided physical measure if and only if μ is an ergodic measure of maximal dimension. Moreover, μ is an ergodic measure of full dimension if and only if* $\dim_H \mathcal{B}(\mu) = \dim_H \Lambda$.

5. MEASURES OF MAXIMAL AND FULL DIMENSION

In this section we study ergodic measures of maximal and full dimension in the case of hyperbolic Hénon maps. First, we consider measures of full dimension. The following result is known (see [12] and [2]). For completeness, we provide a short proof.

Theorem 10. *Let f be hyperbolic volume-preserving Hénon map having maximal entropy. Then $t^u = t^s$, and ν_{t^u} is the unique ergodic measure of full dimension for f.*

Proof: Since $|\det Df| = 1$, Proposition 4 (i) implies that $P^u = P^s$. Thus, by Theorem 5, $t^u = t^s$. Applying Equations (9) and (42), we conclude that $\dim_H \nu_{t^u} = \dim_H \Lambda$, which implies that ν_{t^u} is the unique ergodic measure of full dimension. □

We now consider the volume decreasing case.

Theorem 11. *Let f be a volume decreasing hyperbolic Hénon map having maximal entropy. Then f admits no ergodic measure of full dimension.*

Proof: Assume on the contrary that μ is an ergodic measure of full dimension for f. Then, by Theorem 2, $\mu = \nu_{t^u} = \nu_{t^s}$. On the other hand, Corollary 1 implies $t^s < t^u$. But this contradicts the fact that $\lambda_u(t)$ is strictly decreasing, see Lemma 1. □

We now establish the existence of ergodic measures of maximal dimension. Since a hyperbolic Hénon map *f* having maximal entropy is an Axiom. A surface diffeomorphism with a unique basic set Λ, the existence of an ergodic measure of maximal dimension follows already from work of Barreira and Wolf in [(2)]. However, the proof in [(2)] is technically difficult, and does not provide all information about the measures of maximal dimension. Using that *f* has constant Jacobian, we are able to present in the case of Hénon maps a simplified approach. As before, we start with the volume preserving case.

Corollary 7. *Let f be a volume preserving hyperbolic Hénon map having maximal entropy. Then* ν_{t^u} *is the unique ergodic measure of maximal dimension for f.*

Proof: It follows from Corollary 5 and Theorem 10 that ν_{t^u} is the unique ergodic measure of full dimension of *f*. By definition, every ergodic measure of full dimension is also an ergodic measure of maximal dimension. □

We now consider the volume decreasing case.

Theorem 12. *Let f be a volume decreasing hyperbolic Hénon map having maximal entropy. Then there exists an ergodic measure of maximal dimension for f. If* μ *is an ergodic measure of maximal dimension for f, then* $\mu = \nu_t$ *for some* $t^s < t < t^u$*. Moreover, there are at most finitely many ergodic measures of maximal dimension for f.*

Proof: Let *f* be a volume decreasing hyperbolic Hénon map having maximal entropy. Then, by Corollary 5, $t^s < t^u$. Moreover, it follows from (31) and Proposition 1 (i) that the pressure functions P_u and P_s are not affine. Recall that $\Delta(t) = \dim_H \nu_t$.

Claim 1. There exists $\varepsilon > 0$ such that Δ is strictly increasing on $[0, t^s + \varepsilon)$ and strictly decreasing on $(t^u - \varepsilon, \infty)$.
To prove claim 1, we first notice that Theorem 5 and the fact that $P^{u/s}$ are strictly decreasing functions imply that $P^s(t) > 0$ for all $t \in [0, t^s)$. Analogously, $P^u(t) > 0$ for all $t \in [0, t^u)$. We conclude from Lemma 1, Equation (35) and an elementary continuity argument that there exists $\varepsilon > 0$ such that

$$\frac{d\Delta}{dt} > 0 \tag{64}$$

in $[0, t^s + \varepsilon)$. Therefore, $\Delta(t)$ is strictly increasing in $[0, t^s + \varepsilon)$. Analogously, we can show that there is $\varepsilon > 0$ such that $\Delta(t)$ is strictly decreasing in $(t^u - \varepsilon, \infty)$, which prove the claim.

Claim 1 implies that there is $t_{\max} \in [t^s + \varepsilon, t^u - \varepsilon]$ such that

$$\dim_H \nu_{t_{\max}} = \sup_{t \geq 0} \dim_H \nu_t. \tag{65}$$

Claim 2. The measure $\nu_{t_{\max}}$ is a measure of maximal dimension.
Let $(\nu_k)_{k\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}$ such that

$$\lim_{k\to\infty} \dim_H \nu_k = \delta(f). \tag{66}$$

It follows from (10) that we can assume that all measures ν_k are ergodic. Using claim 1, we may assume that $\dim_H \nu_0 < \dim_H \nu_k$ for all $k \in \mathbb{N}$. Recall that ν_0 is the unique measure of maximal entropy of f. Equation (9) implies that

$$\lambda_u(0) = \lambda_u(\nu_0) > \lambda_u(\nu_k) \tag{67}$$

for all $k \in \mathbb{N}$. Again by claim 1, we may assume that $\dim_H \nu_{t^u} < \dim_H \nu_k$ for all $k \in \mathbb{N}$. It follows from (41) that

$$\frac{h_{\nu_k}(f)}{\lambda_u(\nu_k)} < \frac{h_{\nu_{t^u}}(f)}{\lambda_u(\nu_{t^u})} \tag{68}$$

for all $k \in \mathbb{N}$. Therefore, by (9),

$$\frac{h_{\nu_k}(f)}{\lambda_u(\nu_k) - \log|\det Df|} > \frac{h_{\nu_{t^u}}(f)}{\lambda_u(\nu_{t^u}) - \log|\det Df|} \tag{69}$$

for all $k \in \mathbb{N}$. Equations. (68), (69) imply $h_{\nu_k}(f) > h_{\nu_{t^u}}(f)$, and therefore again by Equation (68) we obtain that

$$\lambda_u(\nu_k) > \lambda_u(\nu_{t^u}) \tag{70}$$

for all $k \in \mathbb{N}$. Since λ_u is a continuous function on $\mathcal{M}$, Equations (67), (70) imply that for all $k \in \mathbb{N}$ there exists $t_k \in (0, t^u)$ such that

$$\lambda_u(\nu_k) = \lambda_u(\nu_{t_k}). \tag{71}$$

Thus, the variational principle (11) implies

$$h_{\nu_k}(f) \leq h_{\nu_{t_k}}(f), \tag{72}$$

hence

$$\dim_H \nu_k \leq \dim_H \nu_{t_k} \tag{73}$$

for all $k \in \mathbb{N}$. This implies

$$\dim_H \nu_k \leq \dim_H \nu_{t_{\max}} \tag{74}$$

for all $k \in \mathbb{N}$. We conclude that $\nu_{t_{\max}}$ is an ergodic measure of maximal dimension.

Claim 3. For every ergodic measure μ of maximal dimension there exists $t^s < t < t^u$ such that $\mu = \nu_t$.

Let μ be an ergodic measure of maximal dimension. We apply to μ (instead of ν_k) the same argumentation as in the proof of claim 2. We obtain that there exists $t \in (0, t^u)$ such that $\lambda_u(\mu) = \lambda_u(\nu_t)$. Since $\dim_H \mu \geq \dim_H \nu_t$, we deduce from equation (9) that $h_\mu(f) \geq h_{\nu_t}(f)$. On the other hand, since ν_t is the equilibrium measure of the potential $-t\phi^u$, we may conclude from (11) and (12) that $h_\mu(f) \leq h_{\nu_t}(f)$. Hence $h_\mu(f) = h_{\nu_t}(f)$. Therefore, the uniqueness of the equilibrium measure of the potential $-t\phi^u$ implies $\mu = \nu_t$. Claim 1 implies that $t \in (t^s, t^u)$, and claim 3 is proven. Finally, since the non-constant real analytic function $t \mapsto \dim_H \nu_t$ can have only finitely many maxima in $[t^s, t^u]$, we conclude that f admits at most finitely many ergodic measures of maximal dimension. □

Remarks.

(i) It follows from Young's formula (9) that the map $\nu \mapsto \dim_H \nu$ is upper semi-continuous on $\mathcal{M}_E$. However, since $\mathcal{M}_E$ is a dense subset of $\mathcal{M}$, see [9] (in particular $\mathcal{M}_E$ is not closed), this does not imply the existence of an ergodic measure of maximal dimension.

(ii) We note that without the assumption of ergodicity there are infinitely many measures of maximal dimension. Namely, if μ is an ergodic measure of maximal dimension, then any measure $\nu \in \mathcal{M}$, which has an ergodic decomposition putting positive measure on μ, is also a measure of maximal dimension. This follows from Theorem 4.

(iii) It follows from Theorems 11 and 12 that if f is non-volume preserving, then f does also not admit a non-ergodic measure of full dimension.

(iv) We would like to point out that we have applied techniques from [25] in the proof of Theorem 12.

The following is a consequence of Corollaries 6 and 7 and Theorem 12.

Corollary 8. *Let f be a hyperbolic Hénon map having maximal entropy. Then f admits at least one and at most finitely many generalized two-sided physical measures. Moreover, if f preserves volume, then f has an unique generalized two-sided physical measure.*

6. GENERALIZED PHYSICAL AND SRB MEASURES FOR HYPERBOLIC SETS ON SURFACES

In this section we consider general hyperbolic sets of surface diffeomorphisms and discuss extensions of, as well as differences to, Theorems 1-3. Furthermore, we study the regular dependence of the dimension of the generalized physical and SRB measures on the diffeomorphism. Let f be a $C^{1+\varepsilon}$-diffeomorphism on a smooth surface M, and let $\Lambda \subset M$ be a locally maximal hyperbolic set of f such that $f|\Lambda$ is topologically mixing. By Λ being hyperbolic we mean that Λ is compact, f-invariant and there exists a continuous Df-invariant splitting $T_\Lambda M = E^u \oplus E^s$ such that $Df|E^u$ is uniformly expanding and $Df|E^s$ is uniformly contracting. We note that this setup naturally occurs in the case of Axiom A diffeomorphisms on compact surfaces. Indeed, by spectral decomposition, the non-wandering set of such a map f can be decomposed into finitely many basic sets, for each of which a certain iterate of f is topologically mixing. Following [(1)] we say that Λ is an attractor if there are arbitrarily small neighborhoods U of Λ such that $f(U) \subset U$. Otherwise, we say that Λ is non-attracting. As mentioned in the introduction physical and SRB measures are well-understood in the case of hyperbolic attractors. Therefore, we focus here on the non-attracting case. To avoid trivialities we also may assume that Λ is of saddle-type (i.e. $\dim E^{u/s} = 1$), because otherwise Λ is simply a repelling periodic orbit. Recall that $\mathcal{M}$ denotes the space of all f-invariant Borel probability measures on Λ, and $\mathcal{M}_E \subset \mathcal{M}$ the subset of ergodic measures (see Section 1.1). We use the definitions of generalized (two-sided) physical and SRB measures on Λ from Section 1.1. Furthermore, we continue to use the notations from Section 2.2 (e.g. $\phi_{u/s}$, $P^{u/s}(t)$, $\lambda_{u/s}(t)$, etc.) for f and Λ. Given $p, q \in \mathbb{R}$ we write $Q(p, q) = P(-p\phi_u + q\phi_s)$, where P denotes the topological pressure of $f|\Lambda$. Hence $P^u(t) = Q(t, 0)$ and $P^s(t) = Q(0, t)$ for all $t \in \mathbb{R}$. Moreover, we denote by $\nu_{p,q}$ the equilibrium measure of the potential $-p\phi_u + q\phi_s$. We note that even though the general properties of the pressure functions discussed in Section 2.2 remain true Propositions 1, 2 do not hold in this more general setup.

6.1. Existence and Uniqueness

We are now in the situation to state our main result concerning the existence and uniqueness of generalized physical measures.

Theorem 13. *Let f be a $C^{1+\varepsilon}$-surface diffeomorphism, and let Λ be a locally maximal non-attracting hyperbolic set of saddle-type such that $f|\Lambda$ is topologically mixing. Then f has a unique generalized physical measure μ^+ on Λ. Moreover, μ^+ is uniquely determined by each of the following properties:*

(i) $\mu^+ = \nu_{t^u,0}$;
(ii) μ^+ *is the unique generalized SRB measure of f on Λ;*
(iii) $\dim_H \mathcal{B}^+(\mu^+) \cap \Lambda = \dim_H \Lambda$;
(iv) $\dim_H \mathcal{B}^+(\mu^+) = \dim_H W^u_\varepsilon(x) \cap \Lambda + 1$.

Proof: The proof is analogous to that in the case of hyperbolic Hénon maps. The only difference occurs in the proof of Theorem 6 where we can apply the fact that the holonomy maps are of class $C^{1+\alpha}$ (see [20]) in order to prove that the map h defined in (46) is bi-Lipschitz continuous. □

Analogously as in the case of hyperbolic Hénon maps Theorem 13 also holds for f^{-1}. We denote the corresponding generalized physical measure by μ^-. We now list several consequences of Theorem 13.

Corollary 9. *Let f, Λ and μ^+ be as in Theorem 13. Then μ^+ is a Gibbs measure.*

Proof: Since μ^+ is the equilibrium measure of the Hölder continuous potential $-t^u\phi_u$ it is a Gibbs measure (see [(1)]). □

Recall that if Λ is a hyperbolic attractor then the equilibrium measure of the potential $-\phi_u$ (i.e. $\nu_{1,0}$) is the unique physical as well as SRB measure for f on Λ. As a consequence of Theorem 13 we obtain that this result, in general, does not hold for non-attracting sets.

Corollary 10. *Let f, Λ and μ^+ be as in Theorem 13. Then the following are equivalent:*

(i) $\mu^+ = \nu_{1,0}$;
(ii) ϕ_u *is cohomologous to a constant.*

Proof: Using that $\mu^+ = \nu_{t^u,0}$ and $t^u < 1$ the result follows from [1, Proposition 4.5]. □

Clearly, ϕ_u not being cohomologous to a constant holds for an open and dense set of maps (with respect to the C^1-topology) in the space of $C^{1+\varepsilon}$ diffeomorphisms on M having a locally maximal non-attracting hyperbolic set. Thus, we obtain the following:

Corollary 11. *$\mu^+(f) \neq \nu_{1,0}(f)$ holds for an open and dense set of maps f with respect to the C^1-topology.*

In order to discuss generalized two-sided physical measures we need the following result concerning the existence of ergodic measures of maximal dimension which was proven in [(2)].

Theorem 14. *Let f be a $C^{1+\varepsilon}$-surface diffeomorphism, and let Λ be a locally maximal hyperbolic set such that $f|\Lambda$ is topologically mixing. Then there exists an ergodic measure of maximal dimension on Λ. Moreover, if μ is an ergodic measure of maximal dimension on Λ then at least one of the following conditions holds:*

(i) *There exits $0 \le p \le t^u$ and $0 \le q \le t^s$ such that $\mu = \nu_{p,q}$;*

(ii) $\lambda_u(\mu) = \min\{\lambda_u(\nu) : \nu \in \mathcal{M}\}$ *and* $\lambda_s(\mu) = \max\{\lambda_s(\nu) : \nu \in \mathcal{M}\}$.

Remark. Rams [18] gave an example of a piecewise linear horseshoe with two distinct ergodic measures of maximal dimension. In fact, it is an open and apparently rather difficult problem to determine whether on hyperbolic sets of surfaces the number of ergodic measures of maximal dimension is in general finite.

We now discuss generalized two-sided physical measures. The following result is a version of Theorem 2 for general hyperbolic sets. The proof is entirely analogous.

Theorem 15. *Let f be a $C^{1+\varepsilon}$-surface diffeomorphism, and let Λ be a locally maximal non-attracting hyperbolic set of saddle-type such that $f|\Lambda$ is topologically mixing. Then f has a generalized two-sided physical measure on Λ. Moreover, μ is a generalized two-sided physical measure if and only if μ is an ergodic measure of maximal dimension.*

Remark. We note that in contrary to the case of hyperbolic Hénon maps we are not able to conclude the finiteness of the set of generalized two-sided physical measures. This is due to the fact that in the general case the ergodic measures of maximal dimension cannot be identified as equilibrium measures of a one-parameter family of potentials (compare Theorems 12 and 14).

The following theorem is an analogue of part (i) of Theorem 3 for general hyperbolic sets.

Theorem 16. *Let f be a $C^{1+\varepsilon}$-surface diffeomorphism, and let Λ be a locally maximal non-attracting hyperbolic set of saddle-type such that $f|\Lambda$ is topologically mixing. Let μ^+, μ^-, μ be as in Theorems 13 and 15. Suppose that $|\det Df(x)| = 1$ holds for all $x \in \Lambda$. Then $\mu = \mu^+ = \mu^-$ and μ is the unique ergodic measure of full dimension.*

Proof: Assume that $|\det Df(x)| = 1$ for all $x \in \Lambda$. Then it follows from (12) that $\nu_{t^u,0} = \nu_{0,t^s}$, and $\nu_{t^u,0}$ is the unique ergodic measure of full dimension. Using that a measure of full dimension is also a measure of maximal dimension and applying Theorems 13, 15 we may conclude that $\mu = \mu^+ = \mu^-$. □

Remark.

(i) Note that Theorem 16 in particular holds if *f* is volume-preserving in a neighborhood of Λ.

(ii) We point out that part (ii) of Theorem 3 has no natural extension to general hyperbolic sets on surfaces. In fact, it is easy to provide examples of piecewise linear horseshoes with $|\det Df| \not\equiv 1$ for which the measures μ^+, μ^- and μ are not pairwise distinct. The problem in the general case is the lack of a cohomology relation between ϕ_u and ϕ_s as it occurs in the case of hyperbolic Hénon maps (see Proposition 1).

6.2. Regular Dependence on the Diffeomorphism

Finally, we discuss the dependence of the dimension of the generalized physical measure on the diffeomorphism. Namely, we are interested as to how the quantities $\dim_H \mathcal{B}^+(\mu^+(f))$ and $\dim_H \mu^+(f)$ vary with f.

More precisely, let f be a C^r-surface diffeomorphism (r large enough), and let Λ be a locally maximal non-attracting hyperbolic set of saddle type of f such that $f|\Lambda$ is topologically mixing. Consider an open neighborhood U of Λ such that $\Lambda = \cap_{n\in\mathbb{Z}} f^n U$. Then (see for example[14]) there exists a neighborhood $\mathcal{U}$ of f in the space of C^r-diffeomorphisms (with respect to the C^r-topology) such that for all $g \in \mathcal{U}$ the set $\Lambda_g = \cap_{n\in\mathbb{Z}} g^n U$ is a locally maximal non-attracting hyperbolic set. Moreover, Λ_g is of saddle type and $g|\Lambda_g$ is topologically mixing. We are interested in the regularity of the maps

$$g \mapsto \dim_H \mathcal{B}^+(\mu^+(g)),\; g \mapsto \dim_H \mu^+(g).$$

It was proven by Mañé in Ref. 15 that the maps $g \mapsto t_g^u$, $g \mapsto t_g^s$ are of class C^{r-1}. Therefore, Theorem 13 and the corresponding version of Theorem 7 for g and Λ_g immediately imply the following:

Corollary 12. *The map $g \mapsto \dim_H \mathcal{B}^+(\mu^+(g))$ is of class C^{r-1} in $\mathcal{U}$.*

Next, we consider the dimension of the generalized physical measure.

Theorem 17. *The map $g \mapsto \dim_H \mu^+(g)$ is of class C^{r-2} in $\mathcal{U}$.*

Proof: Consider the function $\mathcal{Q}: \mathcal{U} \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{Q}(g, p, q) = P(g, -p\phi_u + q\phi_s), \tag{75}$$

where $P(g, \cdot)$ denotes the topological pressure of $g|\Lambda_g$. It follows from work of Mañé in Ref. 15 that $\mathcal{Q}$ is of class C^{r-1} in $\mathcal{U}$. It is proven in [2] that the function $\Delta: \mathcal{U} \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\Delta(g, p, q) = \dim_H \nu_{p,q}(g)$ is of class C^{r-2}. Here $\nu_{p,q}(g)$ denotes the equilibrium measure of $-p\phi_u + q\phi_s$ with respect to $g|\Lambda_g$ (with ϕ_u and ϕ_s defined with respect to g). Therefore, the result follows from Theorem 13 and the fact that $g \mapsto t_g^u$ is of class C^{r-1}. □

We now consider the dependence of the dimension of the generalized two-sided physical measures. Namely, we are interested in the regularity of the map

$$\delta(g) = \sup\{\dim_H \mathcal{B}(\nu) : \nu \in \mathcal{M}_E(g)\}. \tag{76}$$

We note that if $\mu(g)$ is any generalized two-sided physical measure for g on Λ_g then $\delta(g)$ coincides with $\dim_H \mathcal{B}(\mu(g))$. It follows from [(2)] and Theorem 15 that the map δ defined in (76) is continuous. Moreover, if g is close to a volume-preserving map we even obtain a higher regularity.

Theorem 18. *Assume that f is volume-preserving . Then there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of f such that for all $g \in \mathcal{V}$ the map g has a unique generalized two-sided physical measure on Λ_g. Moreover, the map $g \mapsto \delta(g)$ is of class C^{r-3} in $\mathcal{V}$.*

Proof: The result is a consequence of [2, Theorem 11] combined with Theorem 15. □

Finally, we apply our results to the family of hyperbolic Hénon maps. We define the parameter space $\mathcal{H}$ by

$$\mathcal{H} = \{(a, b) \in \mathbb{R}^2 : f_{a,b} \text{is a hyperbolic Hénon with maximal entropy}\}.$$

For $(a, b) \in \mathcal{H}$ let $\mu^+(a, b)$ denote the unique generalized physical measure of $f_{a,b}$ given by Theorem 8. Moreover, let $\delta(f_{a,b})$ be defined as in (76) with respect to $f_{a,b}$. The following result is an immediate consequence of Corollary 12 and Theorems 17, 18.

Corollary 13. *The maps $(a, b) \mapsto \dim_H \mathcal{B}^+(\mu^+(a, b))$, $(a, b) \mapsto \dim_H \mu^+(a, b)$ are real-analytic in $\mathcal{H}$. Moreover, if $(a_0, b_0) \in \mathcal{H}$ such that f_{a_0,b_0} is volume-preserving, then $(a, b) \mapsto \delta(f_{a,b})$ is real analytic in a neighborhood of (a_0, b_0).*

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